

# HOMOLOGICAL METHODS IN COMBINATORIAL COMMUTATIVE ALGEBRA

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ABSTRACT. A Stanley–Reisner ring (equivalently, the quotient of a polynomial ring  $\mathbb{k}[x_1, \dots, x_n]$  by a radical monomial ideal) is a basic but central tool in Combinatorial Commutative Algebra. Almost 20 years ago, I introduced the notion of a *squarefree module*, which is a module version of a Stanley–Reisner ring. This notion makes homological aspects of this area more systematic and deep. This report is a brief survey of the squarefree module theory.

## 1. INTRODUCTION

Let  $S = \mathbb{k}[x_1, \dots, x_n]$  be a polynomial ring over a field  $\mathbb{k}$ , and  $\emptyset \neq \Delta \subseteq 2^{\{1, \dots, n\}}$  a simplicial complex (i.e.,  $F \in \Delta$  and  $G \subseteq F$  imply  $G \in \Delta$ ). We call the quotient ring  $\mathbb{k}[\Delta] := S/I_\Delta$  for  $I_\Delta := (\prod_{i \in F} x_i \mid F \notin \Delta)$  is the *Stanley–Reisner ring* of  $\Delta$ . A Stanley–Reisner ring is one of the most basic notions of Combinatorial Commutative Algebra (see, [3, 11, 15]). Ring theoretic or homological properties of  $\mathbb{k}[\Delta]$  are closely related to the topology of the geometric realization  $|\Delta|$  of  $\Delta$ . For example, we have  $\dim \mathbb{k}[\Delta] = \dim |\Delta| + 1$ , where  $\dim \mathbb{k}[\Delta]$  means the Krull dimension of  $\mathbb{k}[\Delta]$ . Moreover, we have the following.

- Recall the implications of homological (or ring theoretic) properties of noetherian graded commutative  $\mathbb{k}$ -algebras

$$\text{Gorenstein}^* \Rightarrow \text{Gorenstein} \Rightarrow \text{Cohen–Macaulay} \Rightarrow \text{Buchsbaum}.$$

Except the Gorenstein property, the above properties of  $\mathbb{k}[\Delta]$  only depend on the topology of  $|\Delta|$  and  $\text{char}(\mathbb{k})$ .

- If  $X := |\Delta|$  is homeomorphic to a connected manifold (with or without boundary), then  $\mathbb{k}[\Delta]$  is Buchsbaum over any  $\mathbb{k}$ . (The converse is not true. In combinatorial topology, there is the notion of a *homology manifold*. This is still stronger but rather close to the Buchsbaum property of  $\mathbb{k}[\Delta]$ .)

In this case,  $\mathbb{k}[\Delta]$  is Cohen–Macaulay if and only if  $H^i(X; \mathbb{k}) = 0$  for all  $0 < i < \dim X$ . For example, if  $X$  is homeomorphic to a real projective plane, then  $\mathbb{k}[\Delta]$  is Cohen–Macaulay if and only if  $\text{char}(\mathbb{k}) \neq 2$ .

- If  $|\Delta|$  is homeomorphic to a  $d$ -dimensional sphere, then  $\mathbb{k}[\Delta]$  is Gorenstein\* for all  $\mathbb{k}$ . This implies the symmetry of the  $h$ -vector (= a combinatorial data) of a simplicial sphere.

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This report is basically a brief survey of my (our) papers [1, 2, 16, 17, 18]. No version of this report will be submitted for publication elsewhere.

- Unfortunately, the Stanley–Reisner ring theory is NOT compatible with the homotopy theory.

Almost 20 years ago, to apply homological methods to the Stanley–Reisner ring theory more systematically, I introduced the notion of a squarefree module. Since then this notion has been used by several authors in this area. This report is a brief survey of the squarefree module theory. I do not have enough time to introduce the results given by other researchers, and only treat my own results here.

## 2. SQUAREFREE MODULES

Consider the  $\mathbb{N}^n$ -grading  $S = \bigoplus_{\mathbf{a} \in \mathbb{N}^n} S_{\mathbf{a}} = \bigoplus_{\mathbf{a} \in \mathbb{N}^n} \mathbb{k} x^{\mathbf{a}}$ , where  $x^{\mathbf{a}} = \prod_{i=1}^n x_i^{a_i}$  is the monomial with the exponent  $\mathbf{a} = (a_1, \dots, a_n)$ . We denote the graded maximal ideal  $(x_1, \dots, x_n)$  by  $\mathfrak{m}$ . For a  $\mathbb{Z}^n$ -graded module  $M$  and  $\mathbf{a} \in \mathbb{Z}^n$ ,  $M_{\mathbf{a}}$  means the degree  $\mathbf{a}$  component of  $M$ , and  $M(\mathbf{a})$  denotes the shifted module with  $M(\mathbf{a})_{\mathbf{b}} = M_{\mathbf{a}+\mathbf{b}}$ . We denote the category of  $S$ -modules by  $\text{Mod } S$ , and the category of  $\mathbb{Z}^n$ -graded  $S$ -modules by  ${}^*\text{Mod } S$ .

For  $M, N \in {}^*\text{Mod } S$  and  $\mathbf{a} \in \mathbb{Z}^n$ , set  ${}^*\text{Hom}_S(M, N)_{\mathbf{a}} := \text{Hom}_{{}^*\text{Mod } S}(M, N(\mathbf{a}))$ . Then

$${}^*\text{Hom}_S(M, N) := \bigoplus_{\mathbf{a} \in \mathbb{Z}^n} {}^*\text{Hom}_S(M, N)_{\mathbf{a}}$$

has a  $\mathbb{Z}^n$ -graded  $S$ -module structure. If  $M$  is finitely generated, then  ${}^*\text{Hom}_S(M, N)$  is isomorphic to the usual  $\text{Hom}_S(M, N)$  as the underlying  $S$ -module. Thus, we simply denote  ${}^*\text{Hom}_S(M, N)$  by  $\text{Hom}_S(M, N)$  in this case. In the same situation,  $\text{Ext}_S^i(M, N)$  also has a  $\mathbb{Z}^n$ -grading with  $\text{Ext}_S^i(M, N)_{\mathbf{a}} = \text{Ext}_{{}^*\text{Mod } S}^i(M, N(\mathbf{a}))$ .

For  $\mathbf{a} \in \mathbb{Z}^n$ , set  $\text{supp}_+(\mathbf{a}) := \{i \mid a_i > 0\} \subseteq [n] := \{1, \dots, n\}$ . We say  $\mathbf{a} \in \mathbb{Z}^n$  is *squarefree* if  $a_i = 0, 1$  for all  $i \in [n]$ . When  $\mathbf{a} \in \mathbb{Z}^n$  is squarefree, we sometimes identify  $\mathbf{a}$  with  $\text{supp}_+(\mathbf{a})$ .

**Definition 1** ([16]). We say a  $\mathbb{Z}^n$ -graded  $S$ -module  $M$  is *squarefree*, if the following conditions are satisfied.

- (1)  $M$  is  $\mathbb{N}^n$ -graded (i.e.,  $M_{\mathbf{a}} = 0$  if  $\mathbf{a} \notin \mathbb{N}^n$ ), and  $\dim_{\mathbb{k}} M_{\mathbf{a}} < \infty$  for all  $\mathbf{a} \in \mathbb{N}^n$ .
- (2) The multiplication map  $M_{\mathbf{a}} \ni y \mapsto x^{\mathbf{b}}y \in M_{\mathbf{a}+\mathbf{b}}$  is bijective for all  $\mathbf{a}, \mathbf{b} \in \mathbb{N}^n$  with  $\text{supp}_+(\mathbf{a} + \mathbf{b}) = \text{supp}_+(\mathbf{a})$ .

A squarefree module  $M$  is generated by its squarefree part  $\bigcup_{F \subseteq [n]} M_F$  (moreover, it is completely “determined” by its squarefree part). Thus  $M$  is finitely generated. For a simplicial complex  $\Delta \subseteq 2^{[n]}$ ,  $I_{\Delta}$  and  $S/I_{\Delta}$  are squarefree modules. A free module  $S(-F)$ ,  $F \subseteq [n]$ , is also squarefree. In particular, the  $\mathbb{Z}^n$ -graded canonical module  $\omega_S = S(-\mathbf{1})$  of  $S$  is squarefree, where  $\mathbf{1} = (1, \dots, 1)$ .

Let  $\text{Sq } S$  denote the full subcategory of  ${}^*\text{Mod } S$  consisting of all the squarefree modules. Then  $\text{Sq } S$  is closed under kernels, cokernels and extensions in  ${}^*\text{Mod } S$  ([16, Lemma 2.3]).

**Lemma 2** ([17]). For  $M \in \text{Sq } S$ , its  $i$ -th syzygy module  $\text{Syz}_i(M)$  and  $\text{Ext}_S^i(M, \omega_S)$  are also squarefree for all  $i \geq 0$ .

Most  $\mathbb{Z}^n$ -graded  $S$ -modules appearing in the Stanley–Reisner ring theory are squarefree (with or without slight modification).<sup>1</sup>

<sup>1</sup>If we forget  $\mathbb{Z}^n$ -grading, there is an important exception. See the last section of this report.

For a simplicial complex  $\Delta \subsetneq 2^{[n]}$ ,

$$\Delta^\vee := \{ F \subseteq [n] \mid [n] - F \notin \Delta \}$$

is a simplicial complex again. Clearly,  $\Delta^{\vee\vee} = \Delta$ . This is a classical construction called the (combinatorial) *Alexander duality*. In fact,  $|\Delta|$  is contained in the  $(n-2)$ -dimensional sphere  $\mathbf{S}^{n-2} := |2^{[n]} - \{[n]\}|$ , and  $|\Delta^\vee|$  is homotopic to the complement  $\mathbf{S}^{n-2} - |\Delta|$ . This duality is very useful in the Stanley–Reisner ring theory.

As shown by Miller [10] and Römer [13], the Alexander duality  $\Delta \mapsto \Delta^\vee$  can be extended to the exact contravariant functor  $\mathbf{A} : \text{Sq } S \rightarrow \text{Sq } S$ . For  $M \in \text{Sq } S$  and  $F \subseteq [n]$ , the component  $\mathbf{A}(M)_F$  is the  $\mathbb{k}$ -dual of  $M_{F^c}$ , and the multiplication map  $\mathbf{A}(M)_F \ni y \mapsto x_i y \in \mathbf{A}(M)_{F \cup \{i\}}$  for  $i \notin F$  is the  $\mathbb{k}$ -dual of  $M_{F^c - \{i\}} \ni y \mapsto x_i y \in M_{F^c}$ . Here  $F^c := [n] - F$ . We have  $\mathbf{A} \circ \mathbf{A} \cong \text{Id}_{\text{Sq } S}$  and  $\mathbf{A}(\mathbb{k}[\Delta]) \cong I_{\Delta^\vee}$ .

For the study of  $\text{Sq } S$ , the incidence algebra of a finite partially ordered set (*poset*, for short) is very useful. So we now recall basic properties of incidence algebras. Let  $P$  be a finite poset. The incidence algebra  $\Lambda = I(P, \mathbb{k})$  of  $P$  over  $\mathbb{k}$  is the  $\mathbb{k}$ -vector space with a basis  $\{e_{x,y} \mid x, y \in P \text{ with } x \geq y\}$ . The  $\mathbb{k}$ -bilinear multiplication defined by  $e_{x,y} e_{z,w} = \delta_{y,z} e_{x,w}$  makes  $\Lambda$  a finite dimensional associative  $\mathbb{k}$ -algebra. (The usual definition is the opposite ring of our  $\Lambda$ . But we use the above definition for the convenience.) Set  $e_x := e_{x,x}$ . Then  $1 = \sum_{x \in P} e_x$  and  $e_x e_y = \delta_{x,y} e_x$ . We have  $\Lambda \cong \bigoplus_{x \in P} \Lambda e_x$  as a left  $\Lambda$ -module.

Let  $\text{mod } \Lambda$  denote the category of finitely generated left  $\Lambda$ -modules. If  $N \in \text{mod } \Lambda$ , we have  $N = \bigoplus_{x \in P} N_x$  as a  $\mathbb{k}$ -vector space, where  $N_x := e_x N$ . Note that  $e_{x,y} N_y \subseteq N_x$  and  $e_{x,y} N_z = 0$  for  $y \neq z$ . If  $f : N \rightarrow N'$  is a morphism in  $\text{mod } \Lambda$ , then  $f(N_x) \subseteq N'_x$ .

**Proposition 3** ([18, Proposition 2.2]). *Regard the power set  $2^{[n]}$  as a poset by inclusions (i.e.,  $2^{[n]}$  is the Boolean lattice), and set  $\Lambda := I(2^{[n]}, \mathbb{k})$ . Then we have a category equivalence*

$$\text{Sq } S \cong \text{mod } \Lambda.$$

If  $M \in \text{Sq } S$  corresponds to  $N \in \text{mod } \Lambda$ , then we have  $M_F \cong e_F N$  as  $\mathbb{k}$ -vector spaces for  $F \subseteq [n]$ .

The category  $\text{mod } \Lambda$  behaves very nicely. In particular, we can easily describe indecomposable projectives and injectives. The next corollary is given by these descriptions. In the sequel, for  $F \subseteq [n]$ , set  $\mathbb{k}[F] := S/(x_i \mid i \notin F) \cong \mathbb{k}[x_i \mid i \in F]$ .

**Corollary 4** ([17]). *Sq } S is an abelian category with enough projectives and injectives. An indecomposable projective (resp. injective) object in Sq } S is isomorphic to  $S(-F)$  (resp.  $\mathbb{k}[F]$ ) for some  $F \subseteq [n]$ . For any squarefree module  $M$ , both  $\text{pd-dim}_{\text{Sq } S} M$  and  $\text{inj-dim}_{\text{Sq } S} M$  are at most  $n$ .*

*Moreover, a simple object in Sq } S is isomorphic to the canonical module  $\omega_{\mathbb{k}[F]} := \text{Ext}_S^{n-\#F}(\mathbb{k}[F], \omega_S)$  of  $\mathbb{k}[F]$  for  $F \subseteq [n]$ . (In fact,  $[\omega_{\mathbb{k}[F]}]_F \cong \mathbb{k}$  and  $[\omega_{\mathbb{k}[F]}]_G = 0$  for all  $G \subseteq [n]$  with  $G \neq F$ .)*

Since  $S(-F)$  is also projective in  ${}^* \text{Mod } S$ , the minimal projective resolution of  $M \in \text{Sq } S$  in the category  $\text{Sq } S$  is also the minimal projective (or free) resolution in  ${}^* \text{Mod } S$ . On the other hand,  $\mathbb{k}[F]$  is not injective in  ${}^* \text{Mod } S$  (an injective object in  ${}^* \text{Mod } S$  is *not* finitely generated as an  $S$ -module), and the relation between the injective resolution in  $\text{Sq } S$  and

that in  ${}^*\text{Mod } S$  is difficult in general. However, the canonical module  $\omega_S$  behaves well in this point of view. The minimal injective resolution of  $\omega_S \in \text{Sq } S$  in the category  $\text{Sq } S$  is the following form

$$(2.1) \quad I^\bullet : 0 \longrightarrow \omega_S \longrightarrow I^0 \longrightarrow \cdots \longrightarrow I^n \longrightarrow 0,$$

$$I^i = \bigoplus_{\substack{F \subseteq [n] \\ \#F = n-i}} \mathbb{k}[F],$$

and the differential map is the sum of the maps of  $\pm\pi : \mathbb{k}[F] \rightarrow \mathbb{k}[F - \{j\}]$  for  $j \in F$ , where  $\pi$  is the canonical surjection.

Let  ${}^*D^\bullet$  and  $D^\bullet$  be the minimal injective resolutions of  $\omega_S$  in the categories  ${}^*\text{Mod } S$  and  $\text{Mod } S$ , respectively. Then  ${}^*D^\bullet$  is a  $\mathbb{Z}^n$ -graded dualizing complex of  $S$ , and  $D^\bullet$  is a usual (non-graded) dualizing complex. (More precisely, it is more natural to consider the translations  ${}^*D^\bullet[n]$  and  $D^\bullet[n]$ . However, we do not care this point here.) It is well-known that there is a quasi-isomorphism  ${}^*D^\bullet \rightarrow D^\bullet$ .

**Lemma 5.** *There is a  $\mathbb{Z}^n$ -graded quasi-isomorphism  $I^\bullet \rightarrow {}^*D^\bullet$ .*

By Proposition 3 and Corollary 4, the derived category  $D^b(\text{Sq } S)$  ( $\cong D_{\text{Sq } S}^b({}^*\text{Mod } S)$ ) behaves well.

**Theorem 6** ([18, §3]).  $\mathbf{D}(-) = \text{RHom}_S(-, \omega_S)$  gives a contravariant functor from  $D^b(\text{Sq } S)$  to itself satisfying  $\mathbf{D} \circ \mathbf{D} \cong \text{Id}$ . For  $M \in \text{Sq } S$ ,  $\mathbf{D}(M)$  is isomorphic to the complex

$$(2.2) \quad I^\bullet(M) : 0 \longrightarrow I^0(M) \longrightarrow I^1(M) \longrightarrow \cdots \longrightarrow I^n(M) \longrightarrow 0,$$

$$I^i(M) = \bigoplus_{\substack{F \subseteq [n] \\ \#F = n-i}} (M_F)^* \otimes_{\mathbb{k}} \mathbb{k}[F]$$

in  $D^b(\text{Sq } S)$ . Here  $(M_F)^*$  is the dual  $\mathbb{k}$ -vector space of  $M_F$ , but its degree is  $0 \in \mathbb{Z}^n$ . The differential is composed of the maps

$$\pm v_j^* \otimes_{\mathbb{k}} \pi : (M_F)^* \otimes_{\mathbb{k}} \mathbb{k}[F] \rightarrow (M_{F-\{j\}})^* \otimes_{\mathbb{k}} \mathbb{k}[F - \{j\}]$$

for  $j \in F$ , where  $v_j^*$  is the  $\mathbb{k}$ -dual of the multiplication map  $v_j : M_{F-\{j\}} \ni y \mapsto x_j y \in M_F$  and  $\pi$  is the natural surjection  $\mathbb{k}[F] \rightarrow \mathbb{k}[F - \{j\}]$ .

The latter half of the theorem is very useful in the explicit computation. The local cohomology module  $H_{\mathfrak{m}}^i(\mathbb{k}[\Delta])$ , which is the graded  $\mathbb{k}$ -dual of

$$H^{n-i}(\mathbf{D}(\mathbb{k}[\Delta])) \cong \text{Ext}_S^{n-i}(\mathbb{k}[\Delta], \omega_S)$$

is important in the Stanley–Reisner ring theory. Applying the above theorem to the case  $M = \mathbb{k}[\Delta]$ , we get the Hochster’s formula for the Hilbert series of  $H_{\mathfrak{m}}^i(\mathbb{k}[\Delta])$  ([3, Theorem 5.3.8]).

**Theorem 7** ([18, Theorem 3.10]). *We have*

$$(\mathbf{A} \circ \mathbf{D})^3 \cong T^n,$$

where  $T$  is the translation functor of  $D^b(\text{Sq } S)$ .

**Example 8.** For  $F \subseteq [n]$ , we have the following.

$$\begin{aligned}
& \mathbf{A} \circ \mathbf{D} \circ \mathbf{A} \circ \mathbf{D} \circ \mathbf{A} \circ \mathbf{D} (S(-F)) \\
&= \mathbf{A} \circ \mathbf{D} \circ \mathbf{A} \circ \mathbf{D} \circ \mathbf{A} (S(-F^c)) \\
&= \mathbf{A} \circ \mathbf{D} \circ \mathbf{A} \circ \mathbf{D} (\mathbb{k}[F]) \\
&= \mathbf{A} \circ \mathbf{D} \circ \mathbf{A} (\omega_{\mathbb{k}[F]})[\#F - n] \\
&= \mathbf{A} \circ \mathbf{D} (\omega_{\mathbb{k}[F^c]})[n - \#F] \\
&= \mathbf{A} (\mathbb{k}[F^c])[\#F^c - n - (n - \#F)] \\
&= \mathbf{A} (\mathbb{k}[F^c])[-n] \\
&= S(-F)[n].
\end{aligned}$$

*Remark 9.* Let  $\Lambda$  be the incidence algebra of  $2^{[n]}$ . If  $N \in \text{mod } \Lambda$ , then it is well-known that  $\text{Hom}_{\mathbb{k}}(N, \mathbb{k})$  has a right  $\Lambda$ -module (i.e., a left  $\Lambda^{\text{op}}$ -module) structure. But the opposite ring  $\Lambda^{\text{op}}$  of  $\Lambda$  is isomorphic to  $\Lambda$  itself by  $\Lambda^{\text{op}} \ni e_{F,G} \mapsto e_{G^c, F^c} \in \Lambda$ . Thus  $\text{Hom}_{\mathbb{k}}(-, \mathbb{k})$  gives a contravariant functor from  $\text{mod } \Lambda$  to itself. Through the equivalence  $\text{mod } \Lambda \cong \text{Sq } S$  of Proposition 3,  $\text{Hom}_{\mathbb{k}}(-, \mathbb{k})$  corresponds to the Alexander duality  $\mathbf{A}(-)$ .

Similarly,  $\text{RHom}_{\Lambda}(-, \Lambda)$  gives a contravariant functor from  $D^b(\text{mod } \Lambda)$  to itself via the isomorphism  $\Lambda \cong \Lambda^{\text{op}}$ . Through the equivalence  $\text{mod } \Lambda \cong \text{Sq } S$ ,  $\text{RHom}_{\Lambda}(-, \Lambda)$  corresponds to  $\mathbf{D}$ .

Professor Osamu Iyama told me that the functor  $\mathbf{A} \circ \mathbf{D} : D^b(\text{Sq } S) \rightarrow D^b(\text{Sq } S)$  corresponds to the Nakayama functor  $\Lambda^* \otimes_{\Lambda}^L - : D^b(\text{mod } \Lambda) \rightarrow D^b(\text{mod } \Lambda)$ , and hence Theorem 7 corresponds to the fact that  $\Lambda$  has the  $(n/3)$ -Calabi–Yau property. This fact can be shown directly. In fact, for the quiver algebra  $\mathbb{k}A_2$  of the quiver  $\bullet \rightarrow \bullet$ , we have  $\Lambda \cong (\mathbb{k}A_2)^{\otimes n}$  as  $\mathbb{k}$ -algebras. Moreover, it is well-known that  $\mathbb{k}A_2$  has the  $(1/3)$ -Calabi–Yau property<sup>2</sup>, hence  $(\mathbb{k}A_2)^{\otimes n}$  has the  $(n/3)$ -Calabi–Yau property. I do not give a reference on fractional Calabi–Yau property here, but I believe that the participants of this symposium know it better than me.

### 3. CONSTRUCTIBLE SHEAVES ASSOCIATED WITH SQUAREFREE MODULES

If we regard  $2^{[n]}$  as a simplicial complex, it is an  $(n - 1)$ -simplex, and its geometric realization  $B$  is homeomorphic to an  $(n - 1)$ -dimensional ball. For  $F \subseteq [n]$  with  $\#F = d > 0$ ,  $|F|$  denotes the geometric realization  $|2^F| \subseteq B$ . Let  $|F|^{\circ}$  be the interior of  $|F|$ , which is homeomorphic to a  $(d - 1)$ -dimensional open ball. Note that

$$B = \bigcup_{\emptyset \neq F \subseteq [n]} |F|^{\circ}$$

is a regular CW complex. For  $F \subseteq [n]$ ,

$$U_F := \bigcup_{F \subseteq G \subseteq [n]} |G|^{\circ}$$

is an open set of  $B$ . Note that  $\{U_F \mid \emptyset \neq F \subseteq [n]\}$  is an open covering of  $B$ .

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<sup>2</sup>More generally,  $\mathbb{k}A_m$  has the  $(m - 1)/(m + 1)$ -Calabi–Yau property.

In [18], from  $M \in \text{Sq } S$ , we constructed a sheaf  $M^+$  on  $B$ . More precisely, the assignment

$$\Gamma(U_F, M^+) = M_F$$

for each  $\emptyset \neq F \subseteq [n]$  and the restriction map

$$\Gamma(U_G, M^+) = M_G \ni y \mapsto x^{F-G} \in M_F = \Gamma(U_F, M^+)$$

for  $\emptyset \neq G \subseteq F \subseteq [n]$  (equivalently,  $U_G \supset U_F$ ) defines a constructible sheaf. Note that  $M_0$  is “irrelevant” to  $M^+$ . For the theory of constructible sheaves (especially, Poincaré-Verdier duality), see [6].

For example,  $\mathbb{k}[\Delta]^+ \cong j_* \mathbb{k}_{|\Delta|}$ , where  $\mathbb{k}_{|\Delta|}$  is the constant sheaf on  $|\Delta|$  with coefficients in  $\mathbb{k}$ , and  $j$  is the embedding map  $|\Delta| \hookrightarrow B$ . Similarly, we have that  $(\omega_S)^+ \cong h_! \mathbb{k}_{B^\circ}$ , where  $\mathbb{k}_{B^\circ}$  is the constant sheaf on the interior  $B^\circ$  of  $B$ , and  $h$  is the embedding map  $B^\circ \hookrightarrow B$ . Note that  $(\omega_S)^+$  is the orientation sheaf of  $B$  with coefficients in  $\mathbb{k}$ .

Let  $\Delta \subseteq 2^{[n]}$  be a simplicial complex, and set  $X := |\Delta| \subseteq B$ . For  $M \in \text{Sq } S$ ,  $M$  is an  $\mathbb{k}[\Delta]$ -module (i.e.,  $\text{ann}(M) \supset I_\Delta$ ) if and only if  $\text{Supp}(M^+) := \{x \in B \mid (M^+)_x \neq 0\} \subseteq X$ . In this case, we have

$$H^i(B; M^+) \cong H^i(X; M^+|_X)$$

for all  $i$ . Here  $M^+|_X$  is the restriction of the sheaf  $M^+$  to the closed set  $X \subseteq B$ .

The following theorems treat the local cohomology module  $H_m^i(M)$  of  $M \in \text{Sq } S$ . Since it is the  $\mathbb{Z}^n$ -graded  $\mathbb{k}$ -dual of the squarefree module  $\text{Ext}_S^{n-i}(M, \omega_S)$ , the “anti-squarefree part”  $\bigcup_{F \subseteq [n]} [H_m^i(M)]_{-F}$  determines  $H_m^i(M)$ .

**Theorem 10** ([18, Theorem 3.3]). *For  $M \in \text{Sq } S$ , we have*

$$H^i(B; M^+) \cong [H_m^{i+1}(M)]_0 \quad \text{for all } i \geq 1,$$

and an exact sequence

$$(3.1) \quad 0 \longrightarrow [H_m^0(M)]_0 \longrightarrow M_0 \longrightarrow H^0(B; M^+) \longrightarrow [H_m^1(M)]_0 \longrightarrow 0.$$

In particular, for  $\Delta \subseteq 2^{[n]}$  with  $X := |\Delta|$ , we have

$$[H_m^{i+1}(\mathbb{k}[\Delta])]_0 \cong \tilde{H}^i(X; \mathbb{k}) \quad \text{for all } i \geq 0,$$

where  $\tilde{H}^i(X; \mathbb{k})$  denotes the  $i$ th reduced cohomology of  $X$  with coefficients in  $\mathbb{k}$ .

**Theorem 11** ([18, Theorem 3.5]). *For  $M \in \text{Sq } S$  and  $\emptyset \neq F \subseteq [n]$ , we have*

$$[H_m^{i+1}(M)]_{-F} \cong H_c^i(U_F, j^* M^+) \quad \text{for all } i \geq 0,$$

where  $H_c^i(-)$  stands for the cohomology with the compact support, and  $j : U_F \hookrightarrow B$  is the embedding map. In particular, for  $\Delta \subseteq 2^{[n]}$  with  $X := |\Delta|$ , we have

$$[H_m^{i+1}(\mathbb{k}[\Delta])]_{-F} \cong H_c^i(X \cap U_F; \mathbb{k}) \quad \text{for all } i \geq 0.$$

Let  $\text{Sh}(B)$  be the category of constructible sheaves on  $B$ . Since the functor  $(-)^+ : \text{Sq } S \rightarrow \text{Sh}(B)$  is exact, it can be extended to  $D^b(\text{Sq } S) \rightarrow D^b(\text{Sh}(B))$ .

On the other hand,  $X := |\Delta| \subseteq B$  admits Verdier’s dualizing complex  $\mathcal{D}_X^\bullet \in D^b(\text{Sh}(X))$  with coefficients in  $\mathbb{k}$ . For example,  $\mathcal{D}_B^\bullet$  is isomorphic to  $(\omega_S)^+[n-1]$  in  $D^b(\text{Sh}(B))$ .

**Theorem 12** ([18, Theorem 4.2]). *Set  $X := |\Delta|$ . If  $M \in \text{Sq } S$  is a  $\mathbb{k}[\Delta]$ -module, then we have  $\text{Supp}(\text{Ext}_R^{n-i}(M, \omega_S)^+) \subseteq X$  and*

$$\text{Ext}_R^{n-i}(M, \omega_S)^+|_X \cong \mathcal{E}xt^{1-i}(M^+|_X, \mathcal{D}_X^\bullet)$$

in  $D^b(\text{Sh}(X))$ .

**Corollary 13** ([18]). *With the above notation,  $(\mathbf{D}(\mathbb{k}[\Delta][n-1]))^+|_X$  is isomorphic to  $\mathcal{D}_X^\bullet$ . Hence, for the complex  $I^\bullet(\mathbb{k}[\Delta])$  given by putting  $M = \mathbb{k}[\Delta]$  to (2.2),  $(I^\bullet(\mathbb{k}[\Delta]))^+|_X$  is isomorphic to  $\mathcal{D}_X^\bullet$ .*

If  $\mathbb{k}[\Delta]$  is Buchsbaum with  $\dim \mathbb{k}[\Delta] = d$ , its canonical module  $\omega_{\mathbb{k}[\Delta]} := \text{Ext}_S^{n-d}(\mathbb{k}[\Delta], \omega_S)$  is very important.

**Corollary 14** ([18]). *Set  $X := |\Delta|$  as above. Then we have the following.*

- (i)  $\mathbb{k}[\Delta]$  is Buchsbaum if and only if  $\mathcal{H}^i(\mathcal{D}_X^\bullet) = 0$  for all  $i \neq -\dim X$ .
- (ii) *If  $X$  is homeomorphic to a connected manifold with or without boundary ( $\Rightarrow \mathbb{k}[\Delta]$  is Buchsbaum), then  $(\omega_{\mathbb{k}[\Delta]})^+|_X$  is the orientation sheaf of  $X$  with coefficients in  $\mathbb{k}$ . In particular, if  $X$  is homeomorphic to an orientable manifold then  $\mathbb{k}[\Delta] \cong \omega_{\mathbb{k}[\Delta]}$  in  ${}^*\text{Mod } S$ .*

#### 4. APPLICATIONS

The purpose of this section is to introduce an example of an application of squarefree modules. While the statements do not contain the term ‘‘squarefree modules’’, it is hard (maybe impossible) to prove them without this notion.

**Definition 15** ([8, 9]). Let  $I \subset S = \mathbb{k}[x_1, \dots, x_n]$  be a graded ideal. Set

$$\lambda_{i,j}(S/I) := \mu^i(\mathfrak{m}, H_I^{n-j}(S)),$$

and call it the  $(i, j)$ -th *Lyubeznik number* of  $S/I$ . Here the right hand side means the  $i$ -th Bass number of the local cohomology module  $H_I^{n-j}(S)$ . (It is known that we always have  $\lambda_{i,j}(S/I) < \infty$ , and  $\lambda_{i,j}(S/I) \neq 0$  implies  $0 \leq i \leq j$ .)

Here I do not explain any background of this notion. However, this is an important and difficult invariant. In general case, even  $\lambda_{i,j}(S/I) < \infty$  is highly nontrivial, and one has to use the  $D$ -module theory to show this.

For  $\lambda_{i,j}(\mathbb{k}[\Delta])$ , we can show the following results using the squarefree module theory. We only remark that  $\lambda_{i,j}(S/I) = \lambda_{i,j}(S/\sqrt{I})$  for a general graded ideal  $I$  by definition, so if one wants to know the Lyubeznik numbers of monomial ideals then it suffices to consider  $\lambda_{i,j}(\mathbb{k}[\Delta])$ . (If  $I \subset S$  is a radical monomial ideal, there is a simplicial complex  $\Delta \subseteq 2^{[n]}$  such that  $I = I_\Delta$ .)

**Theorem 16** ([17, Corollary 3.10]). *We have*

$$\lambda_{i,j}(\mathbb{k}[\Delta]) = \dim_{\mathbb{k}}[\text{Ext}_S^{n-i}(\text{Ext}_S^{n-j}(\mathbb{k}[\Delta], \omega_S), \omega_S)]_0.$$

**Theorem 17** ([1, Theorem 5.3]). *For a simplicial complex  $\Delta \subseteq 2^{[n]}$  and each  $i, j$ ,  $\lambda_{i,j}(\mathbb{k}[\Delta])$  only depends on the topology of  $|\Delta|$  and  $\text{char}(\mathbb{k})$ .*

Since the theory of Stanley–Reisner rings is very rich, it is a natural strategy to reduce problems on general (i.e., non-radical) monomial ideals to those on radical monomial ideals. This is a classical technique called “polarization”.

Set  $\tilde{S} := \mathbb{k}[x_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq d]$ . For  $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$ ,  $x^{\mathbf{a}}$  denotes the monomial  $\prod_{i=1}^n x_i^{a_i} \in S$ . For a monomial  $x^{\mathbf{a}} \in S$  with  $\deg(x^{\mathbf{a}}) \leq d$ , set

$$\text{pol}(x^{\mathbf{a}}) := \prod_{1 \leq i \leq n} x_{i,1} x_{i,2} \cdots x_{i,a_i} \in \tilde{S}.$$

If  $I = (x^{\mathbf{a}_1}, \dots, x^{\mathbf{a}_r}) \subseteq S$  is a monomial ideal with  $\deg(x^{\mathbf{a}_i}) \leq d$  for all  $1 \leq i \leq r$ , we set

$$\text{pol}(I) := (\text{pol}(x^{\mathbf{a}_i}) \mid 1 \leq i \leq r).$$

For example, if  $I = (x_1^2, x_1 x_2^2, x_1 x_2 x_3, x_2^2 x_3)$  then

$$\text{pol}(I) = (x_{1,1} x_{1,2}, x_{1,1} x_{2,1} x_{2,2}, x_{1,1} x_{2,1} x_{3,1}, x_{2,1} x_{2,2} x_{3,1}).$$

Note that

$$\Theta := \{x_{i,1} - x_{i,j} \mid 1 \leq i \leq n, 2 \leq j \leq d\} \subseteq \tilde{S}$$

gives the isomorphism  $\tilde{S}/(\Theta) \cong S$  induced by  $\tilde{S} \ni x_{i,j} \mapsto x_i \in S$ . Then  $\text{pol}(I)$  satisfies the following properties.

- (1) Through the isomorphism  $\tilde{S}/(\Theta) \cong S$ , we have  $\tilde{S}/(\Theta) \otimes_{\tilde{S}} \tilde{S}/\text{pol}(I) \cong S/I$
- (2)  $\Theta$  forms a  $(\tilde{S}/\text{pol}(I))$ -regular sequence.<sup>3</sup>

Set  $\mathbf{d} := (d, d, \dots, d) \in \mathbb{N}^n$ . Then the polarization is extended to a functor from the category of *positively  $\mathbf{d}$ -determined  $S$ -modules* in the sense of Miller [10] to the category  $\text{Sq} \tilde{S}$ . The polarization functor essentially commutes with the canonical module dual  $\text{Ext}_S^i(-, \omega_S)$ . Combining this fact, Theorem 16, and a result in [4], we can show the following.

**Theorem 18** ([2, Theorem 5.1]). *For a monomial ideal  $I \subset S$ , set  $h := \dim(\tilde{S}/\text{pol}(I)) - \dim(S/I) = n(d-1)$ . Then we have*

$$\lambda_{i,j}(S/I) = \lambda_{i+h,j+h}(\tilde{S}/\text{pol}(I)).$$

for every  $i, j \in \mathbb{N}$ .

## 5. FURTHER DISCUSSION

When we introduced the notion of a squarefree module almost 20 years ago, I thought that most important modules appearing in the study of Stanley–Reisner rings are (essentially) squarefree. However, I have changed my opinion.

If  $\dim \mathbb{k}[\Delta] = d$ , there is a set  $\Theta = \{\theta_1, \dots, \theta_d\} \subseteq \mathbb{k}[\Delta]_1$  of linear forms such that  $A_\Theta := \mathbb{k}[\Delta]/(\Theta)$  is artinian. Note that this ring is still  $\mathbb{Z}$ -graded, but no longer  $\mathbb{Z}^n$ -graded (since  $\theta_i$ 's are not monomials). The artinian algebra  $A_\Theta$  is important when  $\mathbb{k}[\Delta]$  is Buchsbaum (e.g.,  $|\Delta|$  is a manifold), especially when  $\mathbb{k}[\Delta]$  is Gorenstein (e.g.,  $|\Delta|$  is a sphere).

In the theory of artinian graded  $\mathbb{k}$ -algebras, the following condition has become very popular in this decade.

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<sup>3</sup>“Regular sequence” is very basic and important notion in commutative algebra.



**Definition 19** ([5]). Let  $A = \bigoplus_{i=0}^c A_i$  be an artinian graded commutative  $\mathbb{k}$ -algebra with  $A_0 = \mathbb{k}$ . We say  $A$  satisfies the (*strong*) *Lefschetz property*, if there is a linear form  $y \in A_1$  such that the multiplication map

$$A_i \ni x \longmapsto y^j x \in A_{i+j}$$

is either injective or surjective for all  $i, j \geq 0$ .

*Remark 20.* Of course, this notion comes from algebraic topology. The cohomology rings of “nice” manifolds have the Lefschetz property. In this case, the ring is Gorenstein by Poincaré duality. However, in abstract setting, even if  $A$  is Gorenstein, it does not have the Lefschetz property in general.

In the historical paper [15], it is shown that if  $\Delta$  comes from a simplicial polytope then  $A_\Theta$  has Lefschetz property for some  $\Theta$  (the proof uses a strong theorem of algebraic geometry). At that time, people thought this idea can be used only in this case. However the ring theoretic study on Lefschetz property is developing now, and has begun to give powerful tools to the study of Stanley–Reisner rings (e.g., [7]). The standard reference of this theory is [5], while it does not treat Stanley–Reisner rings so much.

However, the Lefschetz property is far from compatible with argument using squarefree modules. More precisely, the operation  $\mathbb{k}[\Delta] \mapsto A_\Theta$  has been important from the early stage of the study of Stanley–Reisner rings. Simple arguments using  $A_\Theta$  can be replaced by those using squarefree modules. However, advanced technique such as the Lefschetz property cannot be replaced. So I want to find a nice way to combine them.

Finally, I remark that the relation to the theory of *toric manifolds* is a very exciting topic in the study on Stanley–Reisner rings now (e.g., [12]). Frankly I do not know anything about this movement, but it might be an interesting problem to find a nice way to apply squarefree modules to this direction.

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